2.3 LINEAR EQUATIONS

REVIEW MATERIAL

- Review the definition of linear DEs in (6) and (7) of Section 1.1

INTRODUCTION We continue our quest for solutions of first-order DEs by next examining linear equations. Linear differential equations are an especially "friendly" family of differential equations in that, given a linear equation, whether first order or a higher-order kin, there is always a good possibility that we can find some sort of solution of the equation that we can examine.

A DEFINITION The form of a linear first-order DE was given in (7) of Section 1.1. This form, the case when $n = 1$ in (6) of that section, is reproduced here for convenience.

DEFINITION 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

is said to be a **linear equation** in the dependent variable y.

When $g(x) = 0$, the linear equation (1) is said to be **homogeneous;** otherwise, it is **nonhomogeneous.**

STANDARD FORM By dividing both sides of (1) by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form,** of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x). \tag{2}$$

We seek a solution of (2) on an interval I for which both coefficient functions P and f are continuous.

In the discussion that follows we illustrate a property and a procedure and end up with a formula representing the form that every solution of (2) must have. But more than the formula, the property and the procedure are important, because these two concepts carry over to linear equations of higher order.

THE PROPERTY The differential equation (2) has the property that its solution is the **sum** of the two solutions: $y = y_c + y_p$, where y_c is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0 \tag{3}$$

and y_p is a particular solution of the nonhomogeneous equation (2). To see this, observe that

$$\frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p] = \underbrace{\left[\frac{dy_c}{dx} + P(x)y_c\right]}_{0} + \underbrace{\left[\frac{dy_p}{dx} + P(x)y_p\right]}_{f(x)} = f(x).$$

Now the homogeneous equation (3) is also separable. This fact enables us to find y_c by writing (3) as

$$\frac{dy}{y} + P(x)\,dx = 0$$

and integrating. Solving for y gives $y_c = ce^{-\int P(x)dx}$. For convenience let us write $y_c = cy_1(x)$, where $y_1 = e^{-\int P(x)dx}$. The fact that $dy_1/dx + P(x)y_1 = 0$ will be used next to determine y_p.

THE PROCEDURE We can now find a particular solution of equation (2) by a procedure known as **variation of parameters.** The basic idea here is to find a function u so that $y_p = u(x)y_1(x) = u(x)e^{-\int P(x)dx}$ is a solution of (2). In other words, our assumption for y_p is the same as $y_c = cy_1(x)$ except that c is replaced by the "variable parameter" u. Substituting $y_p = uy_1$ into (2) gives

$$\underbrace{u\frac{dy_1}{dx} + y_1\frac{du}{dx}}_{\text{Product Rule}} + P(x)uy_1 = f(x) \qquad \text{or} \qquad u\underbrace{\left[\frac{dy_1}{dx} + P(x)y_1\right]}_{\text{zero}} + y_1\frac{du}{dx} = f(x)$$

so
$$y_1\frac{du}{dx} = f(x).$$

Separating variables and integrating then gives

$$du = \frac{f(x)}{y_1(x)}\,dx \qquad \text{and} \qquad u = \int \frac{f(x)}{y_1(x)}\,dx.$$

Since $y_1(x) = e^{-\int P(x)dx}$, we see that $1/y_1(x) = e^{\int P(x)dx}$. Therefore

$$y_p = uy_1 = \left(\int \frac{f(x)}{y_1(x)}\,dx\right)e^{-\int P(x)dx} = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx,$$

and
$$y = \underbrace{ce^{-\int P(x)dx}}_{y_c} + \underbrace{e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx.}_{y_p} \tag{4}$$

Hence if (2) has a solution, it must be of form (4). Conversely, it is a straightforward exercise in differentiation to verify that (4) constitutes a one-parameter family of solutions of equation (2).

You should not memorize the formula given in (4). However, you should remember the special term

$$e^{\int P(x)dx} \tag{5}$$

because it is used in an equivalent but easier way of solving (2). If equation (4) is multiplied by (5),

$$e^{\int P(x)dx}y = c + \int e^{\int P(x)dx}f(x)\,dx, \tag{6}$$

and then (6) is differentiated,

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x), \tag{7}$$

we get
$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}f(x). \tag{8}$$

Dividing the last result by $e^{\int P(x)dx}$ gives (2).

METHOD OF SOLUTION The recommended method of solving (2) actually consists of (6)–(8) worked in reverse order. In other words, if (2) is multiplied by (5), we get (8). The left-hand side of (8) is recognized as the derivative of the product of $e^{\int P(x)\,dx}$ and y. This gets us to (7). We then integrate both sides of (7) to get the solution (6). Because we can solve (2) by integration after multiplication by $e^{\int P(x)\,dx}$, we call this function an **integrating factor** for the differential equation. For convenience we summarize these results. We again emphasize that you should not memorize formula (4) but work through the following procedure each time.

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) Put a linear equation of form (1) into the standard form (2).

(*ii*) From the standard form identify $P(x)$ and then find the integrating factor $e^{\int P(x)\,dx}$.

(*iii*) Multiply the standard form of the equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx}f(x).$$

(*iv*) Integrate both sides of this last equation.

EXAMPLE 1 Solving a Homogeneous Linear DE

Solve $\dfrac{dy}{dx} - 3y = 0$.

SOLUTION This linear equation can be solved by separation of variables. Alternatively, since the equation is already in the standard form (2), we see that $P(x) = -3$, and so the integrating factor is $e^{\int(-3)\,dx} = e^{-3x}$. We multiply the equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0 \qquad \text{is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 0.$$

Integrating both sides of the last equation gives $e^{-3x}y = c$. Solving for y gives us the explicit solution $y = ce^{3x}$, $-\infty < x < \infty$. ■

EXAMPLE 2 Solving a Nonhomogeneous Linear DE

Solve $\dfrac{dy}{dx} - 3y = 6$.

SOLUTION The associated homogeneous equation for this DE was solved in Example 1. Again the equation is already in the standard form (2), and the integrating factor is still $e^{\int(-3)\,dx} = e^{-3x}$. This time multiplying the given equation by this factor gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}, \qquad \text{which is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

Integrating both sides of the last equation gives $e^{-3x}y = -2e^{-3x} + c$ or $y = -2 + ce^{3x}$, $-\infty < x < \infty$. ■

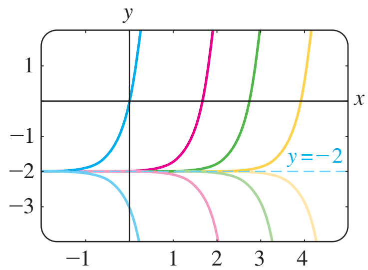

FIGURE 2.3.1 Some solutions of $y' - 3y = 6$

The final solution in Example 2 is the sum of two solutions: $y = y_c + y_p$, where $y_c = ce^{3x}$ is the solution of the homogeneous equation in Example 1 and $y_p = -2$ is a particular solution of the nonhomogeneous equation $y' - 3y = 6$. You need not be concerned about whether a linear first-order equation is homogeneous or nonhomogeneous; when you follow the solution procedure outlined above, a solution of a nonhomogeneous equation necessarily turns out to be $y = y_c + y_p$. However, the distinction between solving a homogeneous DE and solving a nonhomogeneous DE becomes more important in Chapter 4, where we solve linear higher-order equations.

When a_1, a_0, and g in (1) are constants, the differential equation is autonomous. In Example 2 you can verify from the normal form $dy/dx = 3(y + 2)$ that -2 is a critical point and that it is unstable (a repeller). Thus a solution curve with an initial point either above or below the graph of the equilibrium solution $y = -2$ pushes away from this horizontal line as x increases. Figure 2.3.1, obtained with the aid of a graphing utility, shows the graph of $y = -2$ along with some additional solution curves.

CONSTANT OF INTEGRATION Notice that in the general discussion and in Examples 1 and 2 we disregarded a constant of integration in the evaluation of the indefinite integral in the exponent of $e^{\int P(x)dx}$. If you think about the laws of exponents and the fact that the integrating factor multiplies both sides of the differential equation, you should be able to explain why writing $\int P(x)\,dx + c$ is unnecessary. See Problem 44 in Exercises 2.3.

GENERAL SOLUTION Suppose again that the functions P and f in (2) are continuous on a common interval I. In the steps leading to (4) we showed that *if* (2) has a solution on I, then it must be of the form given in (4). Conversely, it is a straightforward exercise in differentiation to verify that any function of the form given in (4) is a solution of the differential equation (2) on I. In other words, (4) is a one-parameter family of solutions of equation (2) and *every solution of* (2) *defined on I is a member of this family.* Therefore we call (4) the **general solution** of the differential equation on the interval I. (See the *Remarks* at the end of Section 1.1.) Now by writing (2) in the normal form $y' = F(x, y)$, we can identify $F(x, y) = -P(x)y + f(x)$ and $\partial F/\partial y = -P(x)$. From the continuity of P and f on the interval I we see that F and $\partial F/\partial y$ are also continuous on I. With Theorem 1.2.1 as our justification, we conclude that there exists one and only one solution of the initial-value problem

$$\frac{dy}{dx} + P(x)y = f(x), \quad y(x_0) = y_0 \tag{9}$$

defined on *some* interval I_0 containing x_0. But when x_0 is in I, finding a solution of (9) is just a matter of finding an appropriate value of c in (4)—that is, to each x_0 in I there corresponds a distinct c. In other words, the interval I_0 of existence and uniqueness in Theorem 1.2.1 for the initial-value problem (9) is the entire interval I.

EXAMPLE 3 General Solution

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x$.

SOLUTION Dividing by x, we get the standard form

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x. \tag{10}$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

$$e^{-4\int dx/x} = e^{\overset{\text{we can use } \ln x \text{ instead of } \ln|x| \text{ since } x>0}{\overset{\downarrow}{-4\ln x}}} = e^{\ln x^{-4}} = x^{-4}.$$

Here we have used the basic identity $b^{\log_b N} = N$, $N > 0$. Now we multiply (10) by x^{-4} and rewrite

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \qquad \text{as} \qquad \frac{d}{dx}[x^{-4}y] = xe^x.$$

It follows from integration by parts that the general solution defined on the interval $(0, \infty)$ is $x^{-4}y = xe^x - e^x + c$ or $y = x^5e^x - x^4e^x + cx^4$. ■

Except in the case in which the lead coefficient is 1, the recasting of equation (1) into the standard form (2) requires division by $a_1(x)$. Values of x for which $a_1(x) = 0$ are called **singular points** of the equation. Singular points are potentially troublesome. Specifically, in (2), if $P(x)$ (formed by dividing $a_0(x)$ by $a_1(x)$) is discontinuous at a point, the discontinuity may carry over to solutions of the differential equation.

EXAMPLE 4 General Solution

Find the general solution of $(x^2 - 9)\dfrac{dy}{dx} + xy = 0$.

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \tag{11}$$

and identify $P(x) = x/(x^2 - 9)$. Although P is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals. On these intervals the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

After multiplying the standard form (11) by this factor, we get

$$\frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0.$$

Integrating both sides of the last equation gives $\sqrt{x^2 - 9}\,y = c$. Thus for either $x > 3$ or $x < -3$ the general solution of the equation is $y = \dfrac{c}{\sqrt{x^2 - 9}}$. ■

Notice in Example 4 that $x = 3$ and $x = -3$ are singular points of the equation and that every function in the general solution $y = c/\sqrt{x^2 - 9}$ is discontinuous at these points. On the other hand, $x = 0$ is a singular point of the differential equation in Example 3, but the general solution $y = x^5e^x - x^4e^x + cx^4$ is noteworthy in that every function in this one-parameter family is continuous at $x = 0$ and is defined on the interval $(-\infty, \infty)$ and not just on $(0, \infty)$, as stated in the solution. However, the family $y = x^5e^x - x^4e^x + cx^4$ defined on $(-\infty, \infty)$ cannot be considered the general solution of the DE, since the singular point $x = 0$ still causes a problem. See Problem 39 in Exercises 2.3.

EXAMPLE 5 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = x, \quad y(0) = 4.$

SOLUTION The equation is in standard form, and $P(x) = 1$ and $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is $e^{\int dx} = e^x$, so integrating

$$\frac{d}{dx}[e^x y] = xe^x$$

gives $e^x y = xe^x - e^x + c$. Solving this last equation for y yields the general solution $y = x - 1 + ce^{-x}$. But from the initial condition we know that $y = 4$ when $x = 0$. Substituting these values into the general solution implies that $c = 5$. Hence the solution of the problem is

$$y = x - 1 + 5e^{-x}, \quad -\infty < x < \infty. \tag{12}$$ ■

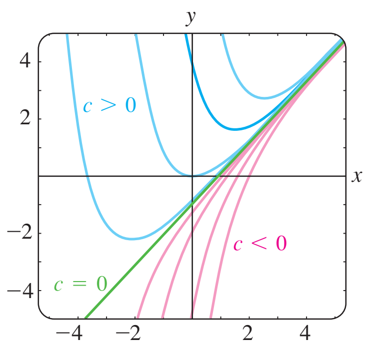

FIGURE 2.3.2 Some solutions of $y' + y = x$

Figure 2.3.2, obtained with the aid of a graphing utility, shows the graph of (12) in dark blue, along with the graphs of other representative solutions in the one-parameter family $y = x - 1 + ce^{-x}$. In this general solution we identify $y_c = ce^{-x}$ and $y_p = x - 1$. It is interesting to observe that as x increases, the graphs of *all* members of the family are close to the graph of the particular solution $y_p = x - 1$, which is shown in solid green in Figure 2.3.2. This is because the contribution of $y_c = ce^{-x}$ to the values of a solution becomes negligible for increasing values of x. We say that $y_c = ce^{-x}$ is a **transient term,** since $y_c \to 0$ as $x \to \infty$. While this behavior is not a characteristic of all general solutions of linear equations (see Example 2), the notion of a transient is often important in applied problems.

DISCONTINUOUS COEFFICIENTS In applications the coefficients $P(x)$ and $f(x)$ in (2) may be piecewise continuous. In the next example $f(x)$ is piecewise continuous on $[0, \infty)$ with a single discontinuity, namely, a (finite) jump discontinuity at $x = 1$. We solve the problem in two parts corresponding to the two intervals over which f is defined. It is then possible to piece together the two solutions at $x = 1$ so that $y(x)$ is continuous on $[0, \infty)$.

EXAMPLE 6 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = f(x), \quad y(0) = 0 \quad$ where $\quad f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$

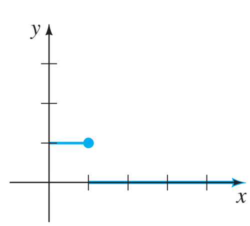

FIGURE 2.3.3 Discontinuous $f(x)$

SOLUTION The graph of the discontinuous function f is shown in Figure 2.3.3. We solve the DE for $y(x)$ first on the interval $[0, 1]$ and then on the interval $(1, \infty)$. For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1 \qquad \text{or, equivalently,} \qquad \frac{d}{dx}[e^x y] = e^x.$$

Integrating this last equation and solving for y gives $y = 1 + c_1 e^{-x}$. Since $y(0) = 0$, we must have $c_1 = -1$, and therefore $y = 1 - e^{-x}$, $0 \le x \le 1$. Then for $x > 1$ the equation

$$\frac{dy}{dx} + y = 0$$

leads to $y = c_2e^{-x}$. Hence we can write

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ c_2e^{-x}, & x > 1. \end{cases}$$

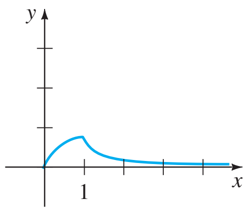

FIGURE 2.3.4 Graph of function in (13)

By appealing to the definition of continuity at a point, it is possible to determine c_2 so that the foregoing function is continuous at $x = 1$. The requirement that $\lim_{x\to 1^+} y(x) = y(1)$ implies that $c_2e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. As seen in Figure 2.3.4, the function

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ (e-1)e^{-x}, & x > 1 \end{cases} \tag{13}$$

is continuous on $(0, \infty)$. ■

It is worthwhile to think about (13) and Figure 2.3.4 a little bit; you are urged to read and answer Problem 42 in Exercises 2.3.

FUNCTIONS DEFINED BY INTEGRALS At the end of Section 2.2 we discussed the fact that some simple continuous functions do not possess antiderivatives that are elementary functions and that integrals of these kinds of functions are called **nonelementary.** For example, you may have seen in calculus that $\int e^{-x^2}\,dx$ and $\int \sin x^2\,dx$ are nonelementary integrals. In applied mathematics some important functions are *defined* in terms of nonelementary integrals. Two such **special functions** are the **error function** and **complementary error function:**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{14}$$

From the known result $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$* we can write $(2/\sqrt{\pi})\int_0^\infty e^{-t^2}\,dt = 1$. Then from $\int_0^\infty = \int_0^x + \int_x^\infty$ it is seen from (14) that the complementary error function $\operatorname{erfc}(x)$ is related to $\operatorname{erf}(x)$ by $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$. Because of its importance in probability, statistics, and applied partial differential equations, the error function has been extensively tabulated. Note that $\operatorname{erf}(0) = 0$ is one obvious function value. Values of $\operatorname{erf}(x)$ can also be found by using a CAS.

EXAMPLE 7 The Error Function

Solve the initial-value problem $\dfrac{dy}{dx} - 2xy = 2, \quad y(0) = 1.$

SOLUTION Since the equation is already in standard form, we see that the integrating factor is $e^{-x^2}\,dx$, so from

$$\frac{d}{dx}[e^{-x^2}y] = 2e^{-x^2} \quad \text{we get} \quad y = 2e^{x^2}\int_0^x e^{-t^2}\,dt + ce^{x^2}. \tag{15}$$

Applying $y(0) = 1$ to the last expression then gives $c = 1$. Hence the solution of the problem is

$$y = 2e^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2} \text{ or } y = e^{x^2}[1 + \sqrt{\pi}\operatorname{erf}(x)].$$

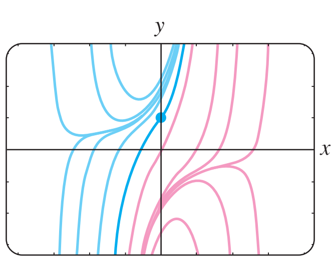

FIGURE 2.3.5 Some solutions of $y' - 2xy = 2$

The graph of this solution on the interval $(-\infty, \infty)$, shown in dark blue in Figure 2.3.5 among other members of the family defined in (15), was obtained with the aid of a computer algebra system. ■

*This result is usually proved in the third semester of calculus.

USE OF COMPUTERS The computer algebra systems *Mathematica* and *Maple* are capable of producing implicit or explicit solutions for some kinds of differential equations using their *dsolve* commands.*

REMARKS

(*i*) In general, a linear DE of any order is said to be homogeneous when $g(x) = 0$ in (6) of Section 1.1. For example, the linear second-order DE $y'' - 2y' + 6y = 0$ is homogeneous. As can be seen in this example and in the special case (3) of this section, the trivial solution $y = 0$ is always a solution of a homogeneous linear DE.

(*ii*) Occasionally, a first-order differential equation is not linear in one variable but is linear in the other variable. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{x + y^2}$$

is not linear in the variable y. But its reciprocal

$$\frac{dx}{dy} = x + y^2 \qquad \text{or} \qquad \frac{dx}{dy} - x = y^2$$

is recognized as linear in the variable x. You should verify that the integrating factor $e^{\int(-1)dy} = e^{-y}$ and integration by parts yield the explicit solution $x = -y^2 - 2y - 2 + ce^y$ for the second equation. This expression is, then, an implicit solution of the first equation.

(*iii*) Mathematicians have adopted as their own certain words from engineering, which they found appropriately descriptive. The word *transient,* used earlier, is one of these terms. In future discussions the words *input* and *output* will occasionally pop up. The function f in (2) is called the **input** or **driving function;** a solution $y(x)$ of the differential equation for a given input is called the **output** or **response.**

(*iv*) The term **special functions** mentioned in conjunction with the error function also applies to the **sine integral function** and the **Fresnel sine integral** introduced in Problems 49 and 50 in Exercises 2.3. "Special Functions" is actually a well-defined branch of mathematics. More special functions are studied in Section 6.3.

*Certain commands have the same spelling, but in *Mathematica* commands begin with a capital letter **(Dsolve),** whereas in *Maple* the same command begins with a lower case letter **(dsolve).** When discussing such common syntax, we compromise and write, for example, *dsolve.* See the *Student Resource and Solutions Manual* for the complete input commands used to solve a linear first-order DE.

EXERCISES 2.3

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{dy}{dx} + 2y = 0$

3. $\dfrac{dy}{dx} + y = e^{3x}$

4. $3\dfrac{dy}{dx} + 12y = 4$

5. $y' + 3x^2y = x^2$

6. $y' + 2xy = x^3$

7. $x^2y' + xy = 1$

8. $y' = 2y + x^2 + 5$

9. $x\dfrac{dy}{dx} - y = x^2 \sin x$

10. $x\dfrac{dy}{dx} + 2y = 3$

11. $x\dfrac{dy}{dx} + 4y = x^3 - x$

12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$

13. $x^2y' + x(x + 2)y = e^x$

14. $xy' + (1 + x)y = e^{-x} \sin 2x$

15. $y\,dx - 4(x + y^6)\,dy = 0$

16. $y\,dx = (ye^y - 2x)\,dy$

17. $\cos x \dfrac{dy}{dx} + (\sin x)y = 1$

18. $\cos^2 x \sin x \dfrac{dy}{dx} + (\cos^3 x)y = 1$

19. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$

20. $(x + 2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy$

21. $\dfrac{dr}{d\theta} + r \sec \theta = \cos \theta$

22. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

In Problems 25–30 solve the given initial-value problem. Give the largest interval I over which the solution is defined.

25. $xy' + y = e^x, \quad y(1) = 2$

26. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

27. $L\dfrac{di}{dt} + Ri = E, \quad i(0) = i_0,$
L, R, E, and i_0 constants

28. $\dfrac{dT}{dt} = k(T - T_m); \quad T(0) = T_0,$
k, T_m, and T_0 constants

29. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

30. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

In Problems 31–34 proceed as in Example 6 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

31. $\dfrac{dy}{dx} + 2y = f(x),\ y(0) = 0,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

32. $\dfrac{dy}{dx} + y = f(x),\ y(0) = 1,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

33. $\dfrac{dy}{dx} + 2xy = f(x),\ y(0) = 2,$ where

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

34. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x),\ y(0) = 0,$ where

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x, & x \ge 1 \end{cases}$$

35. Proceed in a manner analogous to Example 6 to solve the initial-value problem $y' + P(x)y = 4x,\ y(0) = 3,$ where

$$P(x) = \begin{cases} 2, & 0 \le x \le 1, \\ -2/x, & x > 1. \end{cases}$$

Use a graphing utility to graph the continuous function $y(x)$.

36. Consider the initial-value problem $y' + e^x y = f(x)$, $y(0) = 1$. Express the solution of the IVP for $x > 0$ as a nonelementary integral when $f(x) = 1$. What is the solution when $f(x) = 0$? When $f(x) = e^x$?

37. Express the solution of the initial-value problem $y' - 2xy = 1,\ y(1) = 1$, in terms of $\operatorname{erf}(x)$.

Discussion Problems

38. Reread the discussion following Example 2. Construct a linear first-order differential equation for which all nonconstant solutions approach the horizontal asymptote $y = 4$ as $x \to \infty$.

39. Reread Example 3 and then discuss, with reference to Theorem 1.2.1, the existence and uniqueness of a solution of the initial-value problem consisting of $xy' - 4y = x^6 e^x$ and the given initial condition.

(a) $y(0) = 0$ **(b)** $y(0) = y_0,\ y_0 > 0$

(c) $y(x_0) = y_0,\ x_0 > 0,\ y_0 > 0$

40. Reread Example 4 and then find the general solution of the differential equation on the interval $(-3, 3)$.

41. Reread the discussion following Example 5. Construct a linear first-order differential equation for which all solutions are asymptotic to the line $y = 3x - 5$ as $x \to \infty$.

42. Reread Example 6 and then discuss why it is technically incorrect to say that the function in (13) is a "solution" of the IVP on the interval $[0, \infty)$.

43. **(a)** Construct a linear first-order differential equation of the form $xy' + a_0(x)y = g(x)$ for which $y_c = c/x^3$ and $y_p = x^3$. Give an interval on which $y = x^3 + c/x^3$ is the general solution of the DE.

(b) Give an initial condition $y(x_0) = y_0$ for the DE found in part (a) so that the solution of the IVP is $y = x^3 - 1/x^3$. Repeat if the solution is